$$N_s' = \frac{\dot{S}_{gen}}{[(2\pi R q'')^2/kT_0^2]} = \frac{1}{\pi Pe^2} + \frac{1}{8\pi} \times \left[\frac{41n^4 + 74n^3 + 56n^2 + 14n + 1}{(n+1)^2(3n+2)(5n+1)}\right] + \frac{1}{2\pi} A B^{n+1} \left(\frac{3n+1}{n}\right)^n \tag{12}$$

$$R_{opt} = \left\{\frac{3(n+1)}{4\pi^2} A \left[\frac{\dot{M}}{\rho\alpha}\right]^2 \left[\frac{\dot{M}}{\rho\pi}\right]^{n+1} \left(\frac{3n+1}{n}\right)\right\}^{1/(3n+5)} \tag{13}$$

Equations (11) and (12) for the circular pipe are displayed graphically in Fig. 2 and exhibit similar behavior to what was discussed earlier in connection with Fig. 1.

Before closing this section, it is worth clarifying the reason why the viscous terms were neglected in the energy equation which yielded the temperature distribution in the duct [Eqs. (5) and (6) for the parallel plate duct], while these terms were retained in the entropy generation Eq. (3). It is well known that the viscous dissipation terms are negligible in the energy equation for most common power-law fluids (Bird et al.[8]). On the other hand, it can be easily shown based on scaling analysis (the details are omitted for brevity) that the order of magnitude of the viscous dissipation terms in the entropy generation equation differs from that in the energy equation by a factor $0(T/\Delta T)$ where T is the scale of the absolute temperature and ΔT is the scale of the temperature differences encountered in the problem of interest. For example, if the absolute temperature is approximately the room temperature and the temperature differences encountered are about 10 K, the above factor is $0(T/\Delta T) \approx 30$. This indicates that the magnitude of the viscous dissipation terms in the entropy equation is 30 times larger than that in the energy equation. Therefore even if the viscous dissipation terms are neglected in the energy equation, they may be retained in the entropy equation.

IV. Conclusions

In this study, a general expression for the entropy generation in forced convection of power-law fluids was obtained. This expression was applied to study the problem of fully developed forced convection in two basic duct geometries: two parallel plates and a circular pipe. It was found that in both cases, the entropy generation per unit duct length increases monotonically with the power-law index. Interestingly, however, the entropy generation per unit length reaches a plateau at small values of the power-law index. An optimum plate spacing and an optimum pipe diameter were defined that minimize the entropy generation in the duct. Based on the findings of this work given the type of the power-law fluid and the operating conditions, a duct can be sized so that it operates while it produces minimum entropy. Alternatively, for a given duct geometry, one can choose from a number of available power-law fluids the one which will yield the least irreversible convective heat transport.

Acknowledgment

This work was done while the first author (S. K. R.) was on study leave from Tata Energy Research Institute, New Delhi, India.

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Boundary Heat Fluxes for Spectral Radiation from a Uniform Temperature Rectangular Medium

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Nomenclature

A_R	=	aspect ratio of rectangular region, d/b
a	=	absorption coefficient of radiating medium; a_λ, spectral absorption coefficient; a_k, value for kth wavelength band
B_0	=	optical length of short side of rectangle, $a \cdot b$; $B_{0,\lambda} = a_\lambda b$; $B_{0,k} = a_k b$
b, d	=	lengths of short (in y direction) and long (in x direction) sides of rectangle
C_2	=	blackbody radiation constant, 14387.69 μm K
$e_{\lambda b}$	=	hemispherical-spectral emissive power of a blackbody
$F_{0-\lambda}$	=	blackbody energy fraction in wavelength region 0 to λ; F_k, value for kth wavelength band
q_l, q_s	=	local heat fluxes along long and short sides of rectangle; $q_{\lambda,l}$, $q_{\lambda,s}$ spectral values
S_n	=	integral function defined in Eq. (2)
T_r	=	absolute temperature of radiating medium
X, Y	=	dimensionless coordinates, x/b, y/b
x, y, z	=	rectangular coordinates
α, θ	=	argument, and integration variable of S_n function
β	=	value of $C_2/\lambda T$
ϵ_l, ϵ_s	=	emittances of rectangular region to local positions along long and short sides; $\epsilon_{k,l}$, $\epsilon_{k,s}$, values in the kth wavelength band; ϵ_λ local spectral emittance
λ	=	wavelength; λ_c, cutoff wavelength

Received Feb. 25, 1991; revision received March 3, 1991; accepted for publication March 19, 1991.

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Subscripts

k = kth wavelength band
l, s = long and short sides of rectangle
1, 2 = short and long wavelength regions in two-band model

Introduction

In the last 10 years there have been an increasing number of studies on radiation from absorbing media in nonplanar geometries. This is the result of having increased computing speed with which to carry out two- and three-dimensional numerical solutions. Since the numerical programs are usually quite complex, it is desirable to have test cases with which to compare sample numerical results. One test case is a two-dimensional radiating medium of rectangular cross section at uniform temperature. The medium is gray, is emitting and absorbing, and is in an enclosure with cold black walls. For this situation a solution was given by Shah[1] for the local radiative heat fluxes along the boundary of the rectangle. The solution is in the form of a double integral of the modified Bessel function of order zero. It was shown by Siegel[2] that the solution could be transformed into a simpler form that is readily evaluated. Detailed results were obtained in Ref. 2 for several optical dimensions of the rectangle, and for various aspect ratios. This provided a very accurate set of boundary flux values with which results from numerical programs can be compared for a gray medium.

The purpose of this note is to show that the solution in Ref. 2 can be readily extended to provide exact results for spectral radiation of a rectangular medium. The analytical solution has a simple form that can be readily evaluated using any convenient computer integration subroutine. Results can be obtained very easily for a region that radiates with different absorption coefficients in several spectral regions. Sample results are given for a medium with two radiation bands. This illustrates how the distribution of wall fluxes changes as a function of the temperature of the medium.

Analysis

A rectangular medium occupies the region $0 \le x \le d$ and $0 \le y \le b$, where $b \le d$, and is long in the z direction so its radiative behavior is two-dimensional. The medium is spectrally emitting and absorbing, and is at uniform temperature T_r. The bounding walls are black, and are at a low temperature so they do not radiate significant energy. It is desired to obtain the distribution along the walls of the local energy flux radiated by the medium to the walls. Conduction and convection are assumed small compared with radiation.

The boundary flux relations for a gray radiating rectangular medium at uniform temperature were obtained in a convenient form by Siegel.[2] The local heat fluxes along the long and short sides of the rectangle ($0 \le X \le A_R$, $0 \le Y \le 1$) are given by

$$q_l(X) = \sigma T_r^4 \{1 - S_1(B_0X) + S_3(B_0X) - S_1[B_0(A_r - X)] + S_3[B_0(A_R - X)] - B_0 \int_{X'=0}^{AR} [S_0(B_0[(X - X')^2 + 1]^{1/2}) - S_2(B_0[(X - X')^2 + 1]^{1/2})] \, dX'\} \tag{1a}$$

$$q_s(Y) = \sigma T_r^4 \{[1 - S_1(B_0Y) + S_3(B_0Y) - S_1[B_0(1 - Y)] + S_3[B_0(1 - Y)] - B_0 \int_{Y'=0}^{1} [S_0(B_0[A_R^2 + (Y - Y')^2]^{1/2}) - S_2(B_0[A_R^2 + (Y - Y')^2]^{1/2})] \, dY'\} \tag{1b}$$

The S_n are integral functions that arise in two-dimensional radiative transfer. They are discussed in detail by Yuen and Wong.[3] The S_n are readily evaluated by numerical integration of the relation.

$$S_n(\alpha) = \frac{2}{\pi} \int_0^{\pi/2} e^{-\alpha/\cos\theta} \cos^{n-1}\theta \, d\theta \tag{2}$$

The factors in braces multiplying the σT_r^4 in Eqs. (1a) and (1b) can be regarded as the local emittances of the gray medium to the boundary. The emittances depend on the optical length of the short side of the rectangle B_0, the local position along the wall, and the rectangle aspect ratio. The fluxes can then be written in the shorter form

$$q_l(X) = \sigma T_r^4 \epsilon_l(B_0, X, A_R) \tag{3a}$$

$$q_s(Y) = \sigma T_r^4 \epsilon_s(B_0, Y, A_R) \tag{3b}$$

For a spectrally emitting medium the $B_0 \to B_{0,\lambda} = a_\lambda b$ is a function of a_λ. Using spectral quantities, Eq. (3) becomes

$$dq_{\lambda,l}(X) = e_{\lambda b}(\lambda, T_r) \, d\lambda \epsilon_{\lambda,l}(B_{0,\lambda}, X, A_R) \tag{4a}$$

$$dq_{\lambda,s}(Y) = e_{\lambda b}(\lambda, T_r) \, d\lambda \epsilon_{\lambda,s}(B_{0,\lambda}, Y, A_R) \tag{4b}$$

By integrating over all wavelengths the total fluxes along the boundary are then given by

$$\frac{q_l(X)}{\sigma T_r^4} = \frac{\int_0^\infty dq_{\lambda,l}(X)}{\sigma T_r^4} = \frac{1}{\sigma T_r^4} \int_0^\infty \epsilon_{\lambda,l}(B_{0,\lambda}, X, A_R) e_{\lambda b}(\lambda, T_r) \, d\lambda \tag{5a}$$

$$\frac{q_s(Y)}{\sigma T_r^4} = \frac{\int_0^\infty dq_{\lambda,s}(Y)}{\sigma T_r^4} = \frac{1}{\sigma T_r^4} \int_0^\infty \epsilon_{\lambda,s}(B_{0,\lambda}, Y, A_R) e_{\lambda b}(\lambda, T_r) \, d\lambda \tag{5b}$$

For a banded approximation the ϵ_λ does not depend on λ within each band. If in the kth band, $\epsilon_\lambda = \epsilon_k$, Eq. (5) can be written as

$$\frac{q_l(X)}{\sigma T_r^4} = \sum_k \epsilon_{k,l}(B_{0,k}, X, A_R) F_k(T_r) \tag{6a}$$

$$\frac{q_s(Y)}{\sigma T_r^4} = \sum_k \epsilon_{k,s}(B_{0,k}, Y, A_R) F_k(T_r) \tag{6b}$$

where $B_{0,k} = a_k b$. The $F_k(T_r)$ is the fraction of blackbody emission at T_r in the wavelength region of the kth band extending from $\lambda_{k,a}$ to $\lambda_{k,b}$

$$F_k(T_r) = F_{\lambda_{k,a} - \lambda_{k,b}} = \int_{\lambda_{k,a}}^{\lambda_{k,b}} \frac{e_{\lambda b}(\lambda, T_r)}{\sigma T_r^4} \, d\lambda \tag{7}$$

For the kth band the fractional blackbody function is obtained as, $F_k = F_{0-\lambda_{k,b}} - F_{0-\lambda_{k,a}}$. For on-line computer evaluation the integrated blackbody function $F_{0-\lambda}$ can be readily evaluated from the equivalent rapidly converging series used in Ref. 4

$$F_{0-\lambda} = \frac{15}{\pi^4} \sum_n \frac{e^{-n\beta}}{n} \left(\beta^3 + \frac{3\beta^2}{n} + \frac{6\beta}{n^2} + \frac{6}{n^3}\right) \tag{8}$$

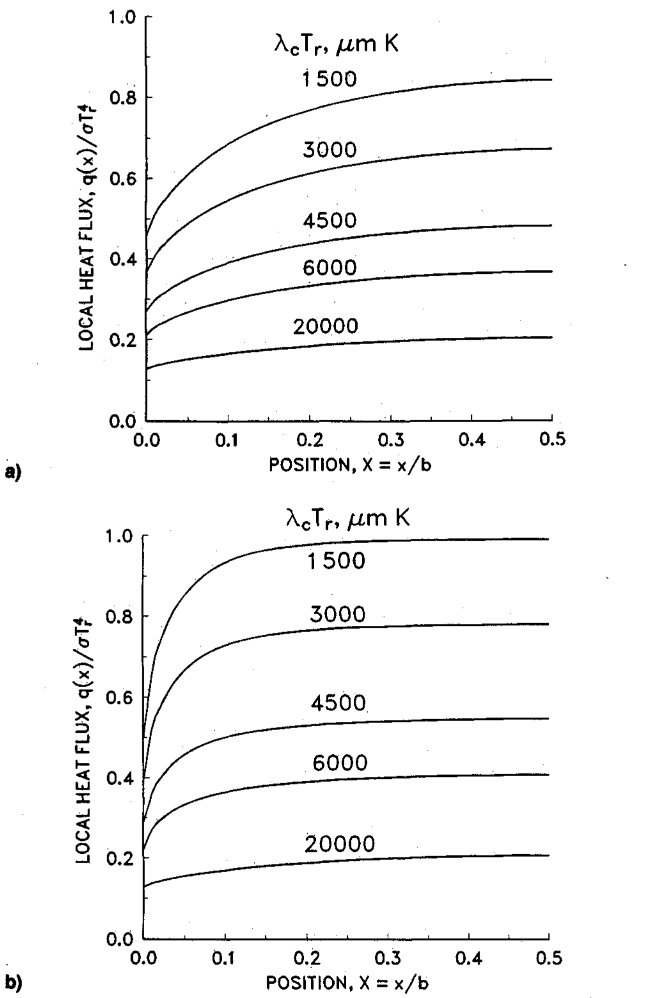

Fig. 1 Dimensionless local radiation heat fluxes received along the sides of a square enclosure. Optical thicknesses: a) $B_{0,1} = 0.2$, $B_{0,2} = 2$; b) $B_{0,1} = 0.2$, $B_{0,2} = 10$.

where $\beta = C_2/\lambda T$ and $C_2 = 14387.69$ μm K.

For the special case of a two-wavelength-band model with band absorption coefficients a_1 and a_2

$$\frac{q_l(X)}{\sigma T_r^4} = \epsilon_{1,l}(B_{0,1}, X, A_R)F_1 + \epsilon_{2,l}(B_{0,2}, X, A_R)F_2 \quad (9a)$$

$$\frac{q_s(Y)}{\sigma T_r^4} = \epsilon_{1,s}(B_{0,1}, Y, A_R)F_1 + \epsilon_{2,s}(B_{0,2}, Y, A_R)F_2 \quad (9b)$$

where $F_1 = F_{0-\lambda_c}$ and $F_2 = 1 - F_{0-\lambda_c}$.

Results and Discussion

To illustrate the results that can be obtained from the spectral analysis, local boundary heat fluxes were evaluated for a square, and for a rectangle with $A_R = 2$. A two-band example was used with absorption coefficient a_1 for $\lambda < \lambda_c$ and a_2 for $\lambda > \lambda_c$. The $a_1 < a_2$ is typical of window materials such as glass and quartz. The β in the fractional blackbody function F depends on λT. Hence a parameter of the two-band results is the product $\lambda_c T_r$. The other parameters are the optical thicknesses $B_{0,1} = a_1 b$ and $B_{0,2} = a_2 b$ in the two spectral bands. Two sets of results were evaluated. For all cases the $B_{0,1} = 0.2$, which is fairly optically thin. For one set of results, $B_{0,2} = 2$, which is intermediate between optically thin and thick. For a second set of results $B_{0,2} = 10$, which is fairly optically thick.

As the parameter $\lambda_c T_r$ is increased, the blackbody spectrum shifts toward shorter wavelengths. Hence for high $\lambda_c T_r$, the heat fluxes are like those for a region with optical dimension $B_{0,1}$, while for $\lambda_c T_r$ the fluxes are like those for a region with dimension $B_{0,2}$. It follows that in Figs. 1 and 2, the lowest curves are very close to the curves in Ref. 2 for $B_0 = 0.2$, while the uppermost curves are like those in Ref. 2 for $B_0 = 2$ or 10. For intermediate values of $\lambda_c T_r$ the energy is partly in each of the two spectral bands. The local heat fluxes vary between the limiting values.

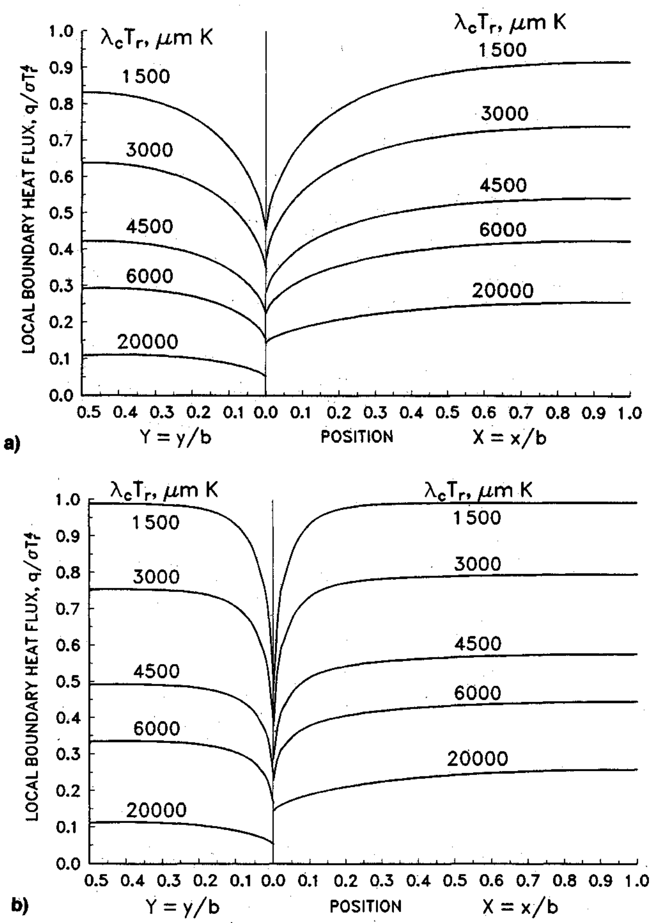

Fig. 2 Dimensionless local radiative heat fluxes received along the sides of a rectangular enclosure of aspect ratio 2. Optical thicknesses: a) $B_{0,1} = 0.2$, $B_{0,2} = 2$; b) $B_{0,1} = 0.2$, $B_{0,2} = 10$.

The results given here are to illustrate how the analytical solution can provide the effect of spectral behavior for radiation in a two-dimensional rectangular geometry. The solution provides exact boundary heat flux values that can be used for comparison with values obtained from general computer programs. The spectral solution given by Eqs. (5) or (6) is easy to evaluate by numerical integration. This can be done for complex variations of the spectral absorption coefficient with wavelength.

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